Homework 9

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Due April 6.

1. Consider complex variables $u_i = q_i + ip_i$ at the points ih, where i takes the values $..., -1, 0, 1, 2, ..., h = 2\pi/n$, n is an integer, and $u_{i+n} = u_i$. Consider the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n}\left[\left(\frac{q_{i+1}-q_i}{h}\right)^2 + \left(\frac{p_{i+1}-p_i}{h}\right)^2 + \frac{1}{2}(q_i^4 + p_i^4)\right].$$

Treat the q, p as conjugate variables; derive the equations of motion. Check formally that as $h \to 0$ these equations converge to the nonlinear Schroedinger equation $iu_t = -u_{xx} + q^3 + ip^3$. Suppose the initial data for the equation are picked from the density $Z^{-1}e^{-H/T}$ for some $T > 0$. By comparing the Hamiltonian with the Feynman-Kac formula in the physicists' notation, deduce that a typical solution of the equation with this kind of data has no derivatives in x (and is therefore a "weak" solution). Check that as $h \to 0$ the Hamiltonian converges to the integral $(1/2)\int_0^{2\pi}(q_x^2 + p_x^2 + (1/2)(q^4 + p^4))dx$.

2. Calculate the magnetization m for the Ising model on a 20^2 lattice, with $T = 0.5$ and with $T = 4$, by Markov Chain Monte-Carlo. Determine the number of steps needed by making sure that the results have converged to something steady.

3. Consider the pdf $f(x) = e^{-x^2}/\sqrt{\pi}$; calculate its entropy. Do the same for the microcanonical density for the Hamiltonian $H = \sum_i p_i^2/2m$, where m is a (constant) mass.

4. Consider a particle with position q and momentum p and Hamiltonian $H = (1/2)(q^2 + p^2)$. Derive the equations of motion and the Liouville equation. Then derive a Fokker-Planck equation for the equations of motion by the methods of Chapter 3 and check that it coincides with the Liouville equation.